AN INTRODUCTION TO LINEAR ALGEBRA

FOR SCIENCE AND ENGINEERING

DANIEL NORMAN • DAN WOLCZUK

THIRD EDITION

An Introduction to Linear Algebra for Science and Engineering

Daniel Norman **Dan Wolczuk**
University of Waterloo

Third Edition

Pearson

Pearson Canada Inc., 26 Prince Andrew Place, North York, Ontario M3C 2H4.

9780134682631

1 20

Library and Archives Canada Cataloguing in Publication

Norman, Daniel, 1938-, author
Introduction to linear algebra for science and engineering / Daniel Norman, Dan Wolczuk, University of Waterloo. – Third edition.

ISBN 978-0-13-468263-1 (softcover)

1. Algebras, Linear–Textbooks. 2. Textbooks. I. Wolczuk, Dan, 1972-, author II. Title.

QA184.2.N67 2018 512'.5 C2018-906600-8

Pearson

Table of Contents

A Note to Students vii

A Note to Instructors x

A Personal Note xv

CHAPTER 1 Euclidean Vector Spaces 1

1.1 Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ 1
1.2 Spanning and Linear Independence in $\mathbb{R}^2$ and $\mathbb{R}^3$ 18
1.3 Length and Angles in $\mathbb{R}^2$ and $\mathbb{R}^3$ 30
1.4 Vectors in $\mathbb{R}^n$ 48
1.5 Dot Products and Projections in $\mathbb{R}^n$ 60
Chapter Review 76

CHAPTER 2 Systems of Linear Equations 79

2.1 Systems of Linear Equations and Elimination 79
2.2 Reduced Row Echelon Form, Rank, and Homogeneous Systems 104
2.3 Application to Spanning and Linear Independence 115
2.4 Applications of Systems of Linear Equations 127
Chapter Review 143

CHAPTER 3 Matrices, Linear Mappings, and Inverses 147

3.1 Operations on Matrices 147
3.2 Matrix Mappings and Linear Mappings 172
3.3 Geometrical Transformations 184
3.4 Special Subspaces 192
3.5 Inverse Matrices and Inverse Mappings 207
3.6 Elementary Matrices 218
3.7 *LU*-Decomposition 226
Chapter Review 232

CHAPTER 4 Vector Spaces 235

4.1 Spaces of Polynomials 235
4.2 Vector Spaces 240
4.3 Bases and Dimensions 249
4.4 Coordinates 264

4.5 General Linear Mappings 273
4.6 Matrix of a Linear Mapping 284
4.7 Isomorphisms of Vector Spaces 297
Chapter Review 304

CHAPTER 5 Determinants 307
5.1 Determinants in Terms of Cofactors 307
5.2 Properties of the Determinant 317
5.3 Inverse by Cofactors, Cramer's Rule 329
5.4 Area, Volume, and the Determinant 337
Chapter Review 343

CHAPTER 6 Eigenvectors and Diagonalization 347
6.1 Eigenvalues and Eigenvectors 347
6.2 Diagonalization 361
6.3 Applications of Diagonalization 369
Chapter Review 380

CHAPTER 7 Inner Products and Projections 383
7.1 Orthogonal Bases in $\mathbb{R}^n$ 383
7.2 Projections and the Gram-Schmidt Procedure 391
7.3 Method of Least Squares 401
7.4 Inner Product Spaces 410
7.5 Fourier Series 417
Chapter Review 422

CHAPTER 8 Symmetric Matrices and Quadratic Forms 425
8.1 Diagonalization of Symmetric Matrices 425
8.2 Quadratic Forms 431
8.3 Graphs of Quadratic Forms 439
8.4 Applications of Quadratic Forms 448
8.5 Singular Value Decomposition 452
Chapter Review 462

CHAPTER 9 Complex Vector Spaces 465
9.1 Complex Numbers 465
9.2 Systems with Complex Numbers 481
9.3 Complex Vector Spaces 486
9.4 Complex Diagonalization 497
9.5 Unitary Diagonalization 500
Chapter Review 505

APPENDIX A **Answers to Mid-Section Exercises** **507**

APPENDIX B **Answers to Practice Problems and Chapter Quizzes** .. **519**

Index **567**

Index of Notations **573**

This page intentionally left blank

A Note to Students

Linear Algebra – What Is It?

Welcome to the third edition of *An Introduction to Linear Algebra for Science and Engineering*! Linear algebra is essentially the study of vectors, matrices, and linear mappings, and is now an extremely important topic in mathematics. Its application and usefulness in a variety of different areas is undeniable. It encompasses technological innovation, economic decision making, industry development, and scientific research. We are literally surrounded by applications of linear algebra.

Most people who have learned linear algebra and calculus believe that the ideas of elementary calculus (such as limits and integrals) are more difficult than those of introductory linear algebra, and that most problems encountered in calculus courses are harder than those found in linear algebra courses. So, at least by this comparison, linear algebra is not hard. Still, some students find learning linear algebra challenging. We think two factors contribute to the difficulty some students have.

First, students do not always see what linear algebra is good for. This is why it is important to read the applications in the text–even if you do not understand them completely. They will give you some sense of where linear algebra fits into the broader picture.

Second, mathematics is often mistakenly seen as a collection of recipes for solving standard problems. Students are often uncomfortable with the fact that linear algebra is "abstract" and includes a lot of "theory." However, students need to realize that there will be no long-term payoff in simply memorizing the recipes–computers carry them out far faster and more accurately than any human. That being said, practicing the procedures on specific examples is often an important step towards a much more important goal: understanding the *concepts* used in linear algebra to formulate and solve problems, and learning to interpret the results of calculations. Such understanding requires us to come to terms with some theory. In this text, when working through the examples and exercises – which are often small – keep in mind that when you do apply these ideas later, you may very well have a million variables and a million equations, but the theory and methods remain constant. For example, Google's PageRank system uses a matrix that has thirty billion columns and thirty billion rows – you do not want to do that by hand! **When you are solving computational problems, always try to observe how your work relates to the theory you have learned.**

Mathematics is useful in so many areas because it is *abstract*: the same good idea can unlock the problems of control engineers, civil engineers, physicists, social scientists, and mathematicians because the idea has been abstracted from a particular setting. One technique solves many problems because someone has established a *theory* of how to deal with these kinds of problems. *Definitions* are the way we try to capture important ideas, and *theorems* are how we summarize useful general facts about the kind of problems we are studying. *Proofs* not only show us that a statement is true; they can help us understand the statement, give us practice using important ideas, and make it easier to learn a given subject. In particular, proofs show us how ideas are tied together, so we do not have to memorize too many disconnected facts.

Many of the concepts introduced in linear algebra are natural and easy, but some may seem unnatural and "technical" to beginners. Do not avoid these seemingly more difficult ideas; use examples and theorems to see how these ideas are an essential part of the story of linear algebra. By learning the "vocabulary" and "grammar" of linear algebra, you will be equipping yourself with concepts and techniques that mathematicians, engineers, and scientists find invaluable for tackling an extraordinarily rich variety of problems.

Linear Algebra – Who Needs It?

Mathematicians

Linear algebra and its applications are a subject of continuing research. Linear algebra is vital to mathematics because it provides essential ideas and tools in areas as diverse as abstract algebra, differential equations, calculus of functions of several variables, differential geometry, functional analysis, and numerical analysis.

Engineers

Suppose you become a control engineer and have to design or upgrade an automatic control system. The system may be controlling a manufacturing process, or perhaps an airplane landing system. You will probably start with a linear model of the system, requiring linear algebra for its solution. To include feedback control, your system must take account of many measurements (for the example of the airplane, position, velocity, pitch, etc.), and it will have to assess this information very rapidly in order to determine the correct control responses. A standard part of such a control system is a Kalman-Bucy filter, which is not so much a piece of hardware as a piece of mathematical machinery for doing the required calculations. Linear algebra is an essential part of the Kalman-Bucy filter.

If you become a structural engineer or a mechanical engineer, you may be concerned with the problem of vibrations in structures or machinery. To understand the problem, you will have to know about eigenvalues and eigenvectors and how they determine the normal modes of oscillation. Eigenvalues and eigenvectors are some of the central topics in linear algebra.

An electrical engineer will need linear algebra to analyze circuits and systems; a civil engineer will need linear algebra to determine internal forces in static structures and to understand principal axes of strain.

In addition to these fairly specific uses, engineers will also find that they need to know linear algebra to understand systems of differential equations and some aspects of the calculus of functions of two or more variables. Moreover, the ideas and techniques of linear algebra are central to numerical techniques for solving problems of heat and fluid flow, which are major concerns in mechanical engineering. Also, the ideas of linear algebra underlie advanced techniques such as Laplace transforms and Fourier analysis.

Physicists

Linear algebra is important in physics, partly for the reasons described above. In addition, it is vital in applications such as the inertia tensor in general rotating motion. Linear algebra is an absolutely essential tool in quantum physics (where, for example, energy levels may be determined as eigenvalues of linear operators) and relativity (where understanding change of coordinates is one of the central issues).

Life and Social Scientists

Input-output models, described by matrices, are often used in economics and other social sciences. Similar ideas can be used in modeling populations where one needs to keep track of sub-populations (generations, for example, or genotypes). In all sciences, statistical analysis of data is of a great importance, and much of this analysis uses linear algebra. For example, the method of least squares (for regression) can be understood in terms of projections in linear algebra.

Managers and Other Professionals

All managers need to make decisions about the best allocation of resources. Enormous amounts of computer time around the world are devoted to linear programming algorithms that solve such allocation problems. In industry, the same sorts of techniques are used in production, networking, and many other areas.

Who needs linear algebra? Almost every mathematician, engineer, scientist, economist, manager, or professional will find linear algebra an important and useful. So, who needs linear algebra? You do!

Will these applications be explained in this book?

Unfortunately, most of these applications require too much specialized background to be included in a first-year linear algebra book. To give you an idea of how some of these concepts are applied, a wide variety of applications are mentioned throughout the text. You will get to see many more applications of linear algebra in your future courses.

How To Make the Most of This Book: SQ3R

The SQ3R reading technique was developed by Francis Robinson to help students read textbooks more effectively. Here is a brief summary of this powerful method for learning. It is easy to learn more about this and other similar strategies online.

Survey: Quickly skim over the section. Make note of any heading or boldface words. Read over the definitions, the statement of theorems, and the statement of examples or exercises (do not read proofs or solutions at this time). Also, briefly examine the figures.

Question: Make a purpose for your reading by writing down general questions about the headings, boldface words, definitions, or theorems that you surveyed. For example, a couple of questions for Section 1.1 could be:
How do we use vectors in $\mathbb{R}^2$ *and* $\mathbb{R}^3$*?*
How does this material relate to what I have previously learned?
What is the relationship between vectors in $\mathbb{R}^2$ *and directed line segments?*
What are the similarities and differences between vectors and lines in $\mathbb{R}^2$ *and in* $\mathbb{R}^3$*?*

Read: Read the material in chunks of about one to two pages. Read carefully and look for the answers to your questions as well as key concepts and supporting details. *Take the time to solve the mid-section exercises before reading past them. Also, try to solve examples before reading the solutions, and try to figure out the proofs before you read them*. If you are not able to solve them, look carefully through the provided solution to figure out the step where you got stuck.

Recall: As you finish each chunk, put the book aside and summarize the important details of what you have just read. Write down the answers to any questions that you made and write down any further questions that you have. Think critically about how well you have understood the concepts, and if necessary, go back and reread a part or do some relevant end of section problems.

Review: This is an ongoing process. Once you complete an entire section, go back and review your notes and questions from the entire section. Test your understanding by trying to solve the end-of-section problems without referring to the book or your notes. Repeat this again when you finish an entire chapter and then again in the future as necessary.

Yes, you are going to find that this makes the reading go much slower for the first couple of chapters. However, students who use this technique consistently report that they feel that they end up spending a lot less time studying for the course as they learn the material so much better at the beginning, which makes future concepts much easier to learn.

A Note to Instructors

Welcome to the third edition of *An Introduction to Linear Algebra for Science and Engineering*! Thanks to the feedback I have received from students and instructors as well as my own research into the science of teaching and learning, I am very excited to present to you this new and improved version of the text. Overall, I believe the modifications I have made complement my overall approach to teaching. I believe in introducing the students slowly to difficult concepts and helping students learn these concepts more deeply by exposing them to the same concepts multiple times over spaced intervals.

One aspect of teaching linear algebra that I find fascinating is that so many different approaches can be used effectively. Typically, the biggest difference between most calculus textbooks is whether they have early or late transcendentals. However, linear algebra textbooks and courses can be done in a wide variety of orders. For example, in China it is not uncommon to begin an introductory linear algebra course with determinants and not cover solving systems of linear equations until after matrices and general vector spaces. Examination of the advantages and disadvantages of a variety of these methods has led me to my current approach.

It is well known that students of linear algebra typically find the computational problems easy but have great difficulty in understanding or applying the abstract concepts and the theory. However, with my approach, I find not only that very few students have trouble with concepts like general vector spaces but that they also retain their mastery of the linear algebra content in their upper year courses.

Although I have found my approach to be very successful with my students, I see the value in a multitude of other ways of organizing an introductory linear algebra course. Therefore, I have tried to write this book to accommodate a variety of orders. See Using This Text To Teach Linear Algebra below.

Changes to the Third Edition

- Some of the content has been reordered to make even better use of the spacing effect. The spacing effect is a well known and extensively studied effect from psychology, which states that students learn concepts better if they are exposed to the same concept multiple times over spaced intervals as opposed to learning it all at once. See:

  Dempster, F.N. (1988). *The spacing effect: A case study in the failure to apply the results of psychological research.* American Psychologist, 43(8), 627–634.

  Fain, R. J., Hieb, J. L., Ralston, P. A., Lyle, K. B. (2015, June), *Can the Spacing Effect Improve the Effectiveness of a Math Intervention Course for Engineering Students?* Paper presented at 2015 ASEE Annual Conference & Exposition, Seattle, Washington.

- The number and type of applications has been greatly increased and are used either to motivate the need for certain concepts or definitions in linear algebra, or to demonstrate how some linear algebra concepts are used in applications.

- A greater emphasis has been placed on the geometry of many concepts. In particular, Chapter 1 has been reorganized to focus on the geometry of linear algebra in $\mathbb{R}^2$ and $\mathbb{R}^3$ before exploring $\mathbb{R}^n$.

- Numerous small changes have been made to improve student comprehension.

Approach and Organization

Students of linear algebra typically have little trouble with computational questions, but they often struggle with abstract concepts and proofs. This is problematic because computers perform the computations in the vast majority of real world applications of linear algebra. Human users, meanwhile, must apply the theory to transform a given problem into a linear algebra context, input the data properly, and interpret the result correctly.

The approach of this book is both to use the spacing effect and to mix theory and computations throughout the course. Additionally, it uses real world applications to both motivate and explain the usefulness of some of the seemingly abstract concepts, and it uses the geometry of linear algebra in $\mathbb{R}^2$ and $\mathbb{R}^3$ to help students visualize many of the concepts. The benefits of this approach are as follows:

- It prevents students from mistaking linear algebra as very easy and very computational early in the course, and then getting overwhelmed by abstract concepts and theories later.

- It allows important linear algebra concepts to be developed and extended more slowly.

- It encourages students to use computational problems to help them understand the theory of linear algebra rather than blindly memorize algorithms.

- It helps students understand the concepts and why they are useful.

One example of this approach is our treatment of the concepts of spanning and linear independence. They are both introduced in Section 1.2 in $\mathbb{R}^2$ and $\mathbb{R}^3$, where they are motivated in a geometrical context. They are expanded to vectors in $\mathbb{R}^n$ in Section 1.4, and used again for matrices in Section 3.1 and polynomials in Section 4.1, before they are finally extended to general vector spaces in Section 4.2.

Other features of the text's organization include

- The idea of linear mappings is introduced early in a geometrical context, and is used to explain aspects of matrix multiplication, matrix inversion, features of systems of linear equations, and the geometry of eigenvalues and eigenvectors. Geometrical transformations provide intuitively satisfying illustrations of important concepts.

- Topics are ordered to give students a chance to work with concepts in a simpler setting before using them in a much more involved or abstract setting. For example, before reaching the definition of a vector space in Section 4.2, students will have seen the ten vector space axioms and the concepts of linear independence and spanning for three different vectors spaces, and will have had some experience in working with bases and dimensions. Thus, instead of being bombarded with new concepts at the introduction of general vector spaces, students will just be generalizing concepts with which they are already familiar.

Pedagogical Features

Since mathematics is best learned by doing, the following pedagogical elements are included in the text:

- A selection of routine mid-section exercises are provided, with answers included in the back of the book. These allow students to use and test their understanding of one concept before moving onto other concepts in the section.
- Practice problems are provided for students at the end of each section. See "A Note on the Exercises and Problems" below.

Applications

Often the applications of linear algebra are not as transparent, concise, or approachable as those of elementary calculus. Most convincing applications of linear algebra require a fairly lengthy buildup of background, which would be inappropriate in a linear algebra text. However, without some of these applications, many students would find it difficult to remain motivated to learn linear algebra. An additional difficultly is that the applications of linear algebra are so varied that there is very little agreement on which applications should be covered.

In this text we briefly discuss a few applications to give students some exposure to how linear algebra is applied.

List of Applications

- Force vectors in physics (Sections 1.1, 1.3)
- Bravais lattice (Section 1.2)
- Graphing quadratic forms (Sections 1.2, 6.2, 8.3)
- Acceleration due to forces (Section 1.3)
- Area and volume (Sections 1.3, 1.5, 5.4)
- Minimum distance from a point to a plane (Section 1.5)
- Best approximation (Section 1.5)
- Forces and moments (Section 2.1)
- Flow through a network (Sections 2.1, 2.4, 3.1)
- Spring-mass systems (Sections 2.4, 3.1, 3.5, 6.1)
- Electrical circuits (Sections 2.4, 9.2)
- Partial fraction decompositions (Section 2.4)
- Balancing chemical equations (Section 2.4)
- Planar trusses (Section 2.4)
- Linear programming (Section 2.4)
- Magic squares (Chapter 4 Review)
- Systems of Linear Difference Equations (Section 6.2)
- Markov processes (Section 6.3)
- Differential equations (Section 6.3)
- Curve of best fit (Section 7.3)

- Overdetermined systems (Section 7.3)
- Fourier series (Section 7.5)
- Small deformations (Sections 6.2, 8.4)
- Inertia tensor (Section 8.4)
- Effective rank (Section 8.5)
- Image compression (Section 8.5)

A wide variety of additional applications are mentioned throughout the text.

A Note on the Exercises and Problems

Most sections contain mid-section exercises. The purpose of these exercises is to give students a way of checking their understanding of some concepts before proceeding to further concepts in the section. Thus, when reading through a chapter, a student should always complete each exercise before continuing to read the rest of the chapter.

At the end of each section, problems are divided into A, B, and C Problems.

The A Problems are practice problems and are intended to provide a sufficient variety and number of standard computational problems and the odd theoretical problem for students to master the techniques of the course; answers are provided at the back of the text. Full solutions are available in the Student Solutions Manual.

The B Problems are homework problems. They are generally identical to the A Problems, with no answers provided, and can be used by by instructors for homework. In a few cases, the B Problems are not exactly parallel to the A Problems.

The C Problems usually require students to work with general cases, to write simple arguments, or to invent examples. These are important aspects of mastering mathematical ideas, and all students should attempt at least some of these–and not get discouraged if they make slow progress. With effort most students will be able to solve many of these problems and will benefit greatly in the understanding of the concepts and connections in doing so.

In addition to the mid-section exercises and end-of-section problems, there is a sample Chapter Quiz in the Chapter Review at the end of each chapter. Students should be aware that their instructors may have a different idea of what constitutes an appropriate test on this material.

At the end of each chapter, there are some Further Problems; these are similar to the C Problems and provide an extended investigation of certain ideas or applications of linear algebra. Further Problems are intended for advanced students who wish to challenge themselves and explore additional concepts.

Using This Text To Teach Linear Algebra

There are many different approaches to teaching linear algebra. Although we suggest covering the chapters in order, the text has been written to try to accommodate a variety of approaches.

Early Vector Spaces We believe that it is very beneficial to introduce general vector spaces immediately after students have gained some experience in working with a few specific examples of vector spaces. Students find it easier to generalize the concepts of spanning, linear independence, bases, dimension, and linear mappings while the earlier specific cases are still fresh in their minds. Additionally, we feel that it can be unhelpful to students to have determinants available too soon. Some students are far too eager to latch onto mindless algorithms involving determinants (for example, to check linear independence of three vectors in three-dimensional space), rather than actually come to terms with the defining ideas. Lastly, this allows eigenvalues, eigenvectors, and diagonalization to be focused on later in the course. I personally find that if diagonalization is taught too soon, students will focus mainly on being able to diagonalize small matrices by hand, which causes the importance of diagonalization to be lost.

Early Systems of Linear Equations For courses that begin with solving systems of linear questions, the first two sections of Chapter 2 may be covered prior to covering Chapter 1 content.

Early Determinants and Diagonalization Some reviewers have commented that they want to be able to cover determinants and diagonalization before abstract vectors spaces and that in some introductory courses abstract vector spaces may be omitted entirely. Thus, this text has been written so that Chapter 5, Chapter 6, most of Chapter 7, and Chapter 8 may be taught prior to Chapter 4 (note that all required information about subspaces, bases, and dimension for diagonalization of matrices over $\mathbb{R}$ is covered in Chapters 1, 2, and 3). Moreover, we have made sure that there is a very natural flow from matrix inverses and elementary matrices at the end of Chapter 3 to determinants in Chapter 5.

Early Complex Numbers Some introductory linear algebra courses include the use of complex numbers from the beginning. We have written Chapter 9 so that the sections of Chapter 9 may be covered immediately after covering the relevant material over $\mathbb{R}$.

A Matrix-Oriented Course For both options above, the text is organized so that sections or subsections involving linear mappings may be omitted without loss of continuity.

MyLab Math

MyLab Math and MathXL are online learning resources available to instructors and students using *An Introduction to Linear Algebra for Science and Engineering*.

MyLab Math provides engaging experiences that personalize, stimulate, and measure learning for each student. MyLab's comprehensive **online gradebook** automatically tracks your students' results on tests, quizzes, homework, and in the study plan. The homework and practice exercises in MyLab Math are correlated to the exercises in the textbook, and MyLab provides **immediate, helpful feedback** when students enter incorrect answers. The **study plan** can be assigned or used for individual practice and is personalized to each student, tracking areas for improvement as students navigate problems. With over 100 questions (all algorithmic) added to the third edition, MyLab Math for *An Introduction to Linear Algebra for Science and Engineering* is a well-equipped resource that can help improve individual students' performance.

To learn more about how MyLab combines proven learning applications with powerful assessment, visit www.pearson.com/mylab or contact your Pearson representative.

A Personal Note

The third edition of *An Introduction to Linear Algebra for Science and Engineering* is meant to engage students and pique their curiosity, as well as provide a template for instructors. I am constantly fascinated by the countless potential applications of linear algebra in everyday life, and I intend for this textbook to be approachable to all. I will not pretend that mathematical prerequisites and previous knowledge are not required. However, the approach taken in this textbook encourages the reader to explore a variety of concepts and provides exposure to an extensive amount of mathematical knowledge. Linear algebra is an exciting discipline. My hope is that those reading this book will share in my enthusiasm.

Acknowledgments

Thanks are expressed to:

Agnieszka Wolczuk for her support and encouragement.

Mike La Croix for all of the amazing figures in the text, and for his assistance in editing, formatting, and LaTeX'ing.

Peiyao Zeng, Daniel Yu, Adam Radek Martinez, Bruno Verdugo Paredes, and Alex Liao for proof-reading and their many valuable comments and suggestions.

Stephen New, Paul McGrath, Ken McCay, Paul Kates, and many other of my colleagues who have helped me become a better instructor.

To all of the reviewers whose comments, corrections, and recommendations have resulted in many positive improvements.

Charlotte Morrison-Reed for all of her hard work in making the third edition of this text possible and for her suggestions and editing.

A very special thank you to Daniel Norman and all those who contributed to the first and second editions.

Dan Wolczuk
University of Waterloo

CHAPTER 1

Euclidean Vector Spaces

CHAPTER OUTLINE

1.1 Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$
1.2 Spanning and Linear Independence in $\mathbb{R}^2$ and $\mathbb{R}^3$
1.3 Length and Angles in $\mathbb{R}^2$ and $\mathbb{R}^3$
1.4 Vectors in $\mathbb{R}^n$
1.5 Dot Products and Projections in $\mathbb{R}^n$

Some of the material in this chapter will be familiar to many students, but some ideas that are introduced here will be new to most. In this chapter we will look at operations on and important concepts related to vectors. We will also look at some applications of vectors in the familiar setting of Euclidean space. Most of these concepts will later be extended to more general settings. A firm understanding of the material from this chapter will help greatly in understanding the topics in the rest of this book.

1.1 Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

We begin by considering the two-dimensional plane in Cartesian coordinates. Choose an origin O and two mutually perpendicular axes, called the x_1-axis and the x_2-axis, as shown in Figure 1.1.1. Any point P in the plane can be uniquely identified by the 2-tuple (p_1, p_2), called the **coordinates** of P. In particular, p_1 is the distance from P to the x_2-axis, with p_1 positive if P is to the right of this axis and negative if P is to the left, and p_2 is the distance from P to the x_1-axis, with p_2 positive if P is above this axis and negative if P is below. You have already learned how to plot graphs of equations in this plane.

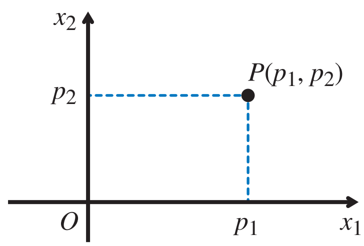


Figure 1.1.1 Coordinates in the plane.

For applications in many areas of mathematics, and in many subjects such as physics, chemistry, economics, and engineering, it is useful to view points more abstractly. In particular, we will view them as **vectors** and provide rules for adding them and multiplying them by constants.

Definition
$\mathbb{R}^2$

We let $\mathbb{R}^2$ denote the set of all vectors of the form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where x_1 and x_2 are real numbers called the **components** of the vector. Mathematically, we write

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1, x_2 \in \mathbb{R} \right\}$$

We say two vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$ are **equal** if $x_1 = y_1$ and $x_2 = y_2$. We write

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Although we are viewing the elements of $\mathbb{R}^2$ as vectors, we can still interpret these geometrically as points. That is, the vector $\vec{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ can be interpreted as the point $P(p_1, p_2)$. Graphically, this is often represented by drawing an arrow from $(0, 0)$ to (p_1, p_2), as shown in Figure 1.1.2. Note, that the point $(0, 0)$ and the points between $(0, 0)$ and (p_1, p_2) should not be thought of as points "on the vector." The representation of a vector as an arrow is particularly common in physics; force and acceleration are vector quantities that can conveniently be represented by an arrow of suitable magnitude and direction.

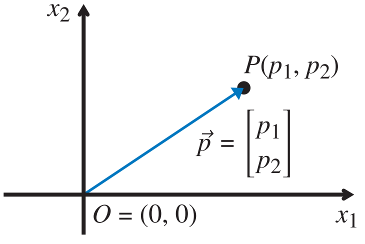


Figure 1.1.2 Graphical representation of a vector.

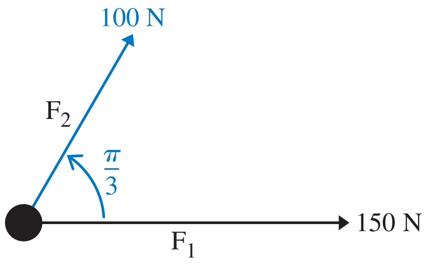
EXAMPLE 1.1.1

An object on a frictionless surface is being pulled by two strings with force and direction as given in the diagram.

(a) Represent each force as a vector in $\mathbb{R}^2$.
(b) Represent the net force being applied to the object as a vector in $\mathbb{R}^2$.

Solution: (a) The force F_1 has 150N of horizontal force and 0N of vertical force. Thus, we can represent this with the vector

$$\vec{F}_1 = \begin{bmatrix} 150 \\ 0 \end{bmatrix}$$



The force F_2 has horizontal component $100\cos\frac{\pi}{3} = 50$ N and vertical component $100\sin\frac{\pi}{3} = 50\sqrt{3}$ N. Therefore, we can represent this with the vector

$$\vec{F}_2 = \begin{bmatrix} 50 \\ 50\sqrt{3} \end{bmatrix}$$

(b) We know from physics that to get the net force we add the horizontal components of the forces together and we add the vertical components of the forces together. Thus, the net horizontal component is $150N + 50N = 200N$. The net vertical force is $0N + 50\sqrt{3}N = 50\sqrt{3}N$. We can represent this as the vector

$$\vec{F} = \begin{bmatrix} 200 \\ 50\sqrt{3} \end{bmatrix}$$

The example shows that in physics we add vectors by adding their corresponding components. Similarly, we find that in physics we multiply a vector by a scalar by multiplying each component of the vector by the scalar.

Since we want our generalized concept of vectors to be able to help us solve physical problems like these and more, we define addition and scalar multiplication of vectors in $\mathbb{R}^2$ to match.

Definition
Addition and Scalar Multiplication in $\mathbb{R}^2$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$. We define **addition** of vectors by

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$$

We define **scalar multiplication** of $\vec{x}$ by a factor of $t \in \mathbb{R}$, called a **scalar**, by

$$t\vec{x} = t\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} tx_1 \\ tx_2 \end{bmatrix}$$

Remark

It is important to note that $\vec{x} - \vec{y}$ is to be interpreted as $\vec{x} + (-1)\vec{y}$.

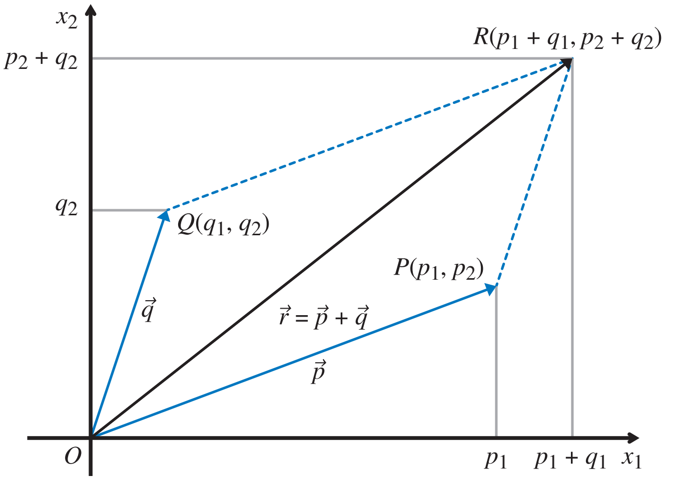


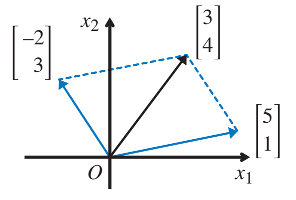
Figure 1.1.3 Addition of vectors $\vec{p}$ and $\vec{q}$.

The addition of two vectors is illustrated in Figure 1.1.3: construct a parallelogram with vectors $\vec{p}$ and $\vec{q}$ as adjacent sides; then $\vec{p} + \vec{q}$ is the vector corresponding to the vertex of the parallelogram opposite to the origin. Observe that the components really are added according to the definition. This is often called the **parallelogram rule for addition**.

EXAMPLE 1.1.2

Let $\vec{x} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \vec{y} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \in \mathbb{R}^2$. Calculate $\vec{x} + \vec{y}$.

Solution: We have $\vec{x} + \vec{y} = \begin{bmatrix} -2+5 \\ 3+1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.



Scalar multiplication is illustrated in Figure 1.1.4. Observe that multiplication by a negative scalar reverses the direction of the vector.

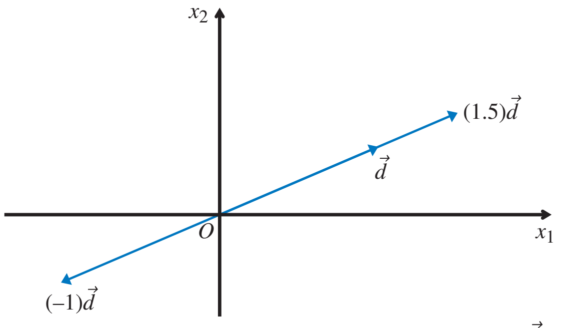


Figure 1.1.4 Scalar multiplication of the vector $\vec{d}$.

EXAMPLE 1.1.3

Let $\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \in \mathbb{R}^2$. Calculate $\vec{u} + \vec{v}$, $3\vec{w}$, and $2\vec{v} - \vec{w}$.

Solution: We get

$$\vec{u} + \vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$3\vec{w} = 3\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

$$2\vec{v} - \vec{w} = 2\begin{bmatrix} -2 \\ 3 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 7 \end{bmatrix}$$

EXERCISE 1.1.1

Let $\vec{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^2$. Calculate each of the following and illustrate with a sketch.

(a) $\vec{u} + \vec{w}$ (b) $-\vec{v}$ (c) $(\vec{u} + \vec{w}) - \vec{v}$

We will frequently look at sums of scalar multiples of vectors. So, we make the following definition.

Definition
Linear Combination

Let $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^2$ and $c_1, \dots, c_k \in \mathbb{R}$. We call the sum $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$ a **linear combination** of the vectors $\vec{v}_1, \dots, \vec{v}_k$.

It is important to observe that $\mathbb{R}^2$ has the property that any linear combination of vectors in $\mathbb{R}^2$ is a vector in $\mathbb{R}^2$ (combining properties V1, V6 in Theorem 1.1.1 below). Although this property is clear for $\mathbb{R}^2$, it does not hold for most subsets of $\mathbb{R}^2$. As we will see in Section 1.4, in linear algebra, we are mostly interested in sets that have this property.

Theorem 1.1.1

For all $\vec{w}, \vec{x}, \vec{y} \in \mathbb{R}^2$ and $s, t \in \mathbb{R}$ we have

V1 $\vec{x} + \vec{y} \in \mathbb{R}^2$ (closed under addition)
V2 $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (addition is commutative)
V3 $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$ (addition is associative)
V4 There exists a vector $\vec{0} \in \mathbb{R}^2$ such that $\vec{z} + \vec{0} = \vec{z}$ for all $\vec{z} \in \mathbb{R}^2$ (zero vector)
V5 For each $\vec{x} \in \mathbb{R}^2$ there exists a vector $-\vec{x} \in \mathbb{R}^2$ such that $\vec{x} + (-\vec{x}) = \vec{0}$ (additive inverses)
V6 $s\vec{x} \in \mathbb{R}^2$ (closed under scalar multiplication)
V7 $s(t\vec{x}) = (st)\vec{x}$ (scalar multiplication is associative)
V8 $(s + t)\vec{x} = s\vec{x} + t\vec{x}$ (a distributive law)
V9 $s(\vec{x} + \vec{y}) = s\vec{x} + s\vec{y}$ (another distributive law)
V10 $1\vec{x} = \vec{x}$ (scalar multiplicative identity)

Observe that the zero vector from property V4 is the vector $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and the additive inverse of $\vec{x}$ from V5 is $-\vec{x} = (-1)\vec{x}$.

The Vector Equation of a Line in $\mathbb{R}^2$

In Figure 1.1.4, it is apparent that the set of all multiples of a non-zero vector $\vec{d}$ creates a line through the origin. We make this our definition of a line in $\mathbb{R}^2$: a **line through the origin in** $\mathbb{R}^2$ is a set of the form

$$\{t\vec{d} \mid t \in \mathbb{R}\}$$

Often we do not use formal set notation but simply write a **vector equation** of the line:

$$\vec{x} = t\vec{d}, \quad t \in \mathbb{R}$$

The non-zero vector $\vec{d}$ is called a **direction vector** of the line.

Similarly, we define a **line through** $\vec{p}$ **with direction vector** $\vec{d} \neq \vec{0}$ to be the set

$$\{\vec{p} + t\vec{d} \mid t \in \mathbb{R}\}$$

which has vector equation

$$\vec{x} = \vec{p} + t\vec{d}, \quad t \in \mathbb{R}$$

This line is parallel to the line with equation $\vec{x} = t\vec{d}, t \in \mathbb{R}$ because of the parallelogram rule for addition. As shown in Figure 1.1.5, each point on the line through $\vec{p}$ can be obtained from a corresponding point on the line $\vec{x} = t\vec{d}, t \in \mathbb{R}$ by adding the vector $\vec{p}$. We say that the line has been **translated** by $\vec{p}$. More generally, two lines are parallel if the direction vector of one line is a non-zero scalar multiple of the direction vector of the other line.

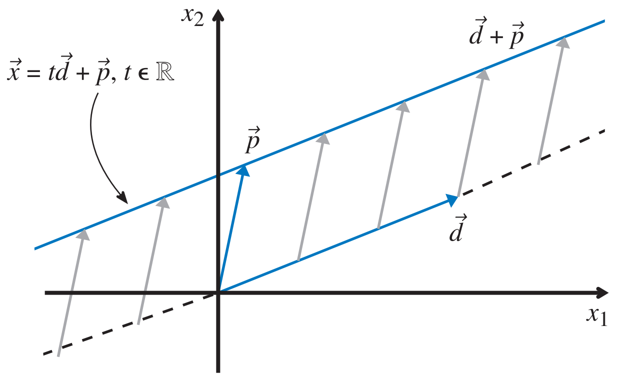


Figure 1.1.5 The line with vector equation $\vec{x} = t\vec{d} + \vec{p}, t \in \mathbb{R}$.

EXAMPLE 1.1.4

A vector equation of the line through the point $P(2, -3)$ with direction vector $\begin{bmatrix} -4 \\ 5 \end{bmatrix}$ is

$$\vec{x} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} + t\begin{bmatrix} -4 \\ 5 \end{bmatrix}, \quad t \in \mathbb{R}$$

EXAMPLE 1.1.5

Write a vector equation of the line through $P(1,2)$ parallel to the line with vector equation

$$\vec{x} = t\begin{bmatrix}2\\3\end{bmatrix}, \quad t \in \mathbb{R}$$

Solution: Since they are parallel, we can choose the same direction vector. Hence, a vector equation of the line is

$$\vec{x} = \begin{bmatrix}1\\2\end{bmatrix} + t\begin{bmatrix}2\\3\end{bmatrix}, \quad t \in \mathbb{R}$$

EXERCISE 1.1.2

Write a vector equation of a line through $P(0,0)$ parallel to the line

$$\vec{x} = \begin{bmatrix}4\\-3\end{bmatrix} + t\begin{bmatrix}3\\-1\end{bmatrix}, \quad t \in \mathbb{R}$$

Sometimes the components of a vector equation are written separately. In particular, expanding a vector equation $\vec{x} = \vec{p} + t\vec{d}, t \in \mathbb{R}$ we get

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}p_1\\p_2\end{bmatrix} + t\begin{bmatrix}d_1\\d_2\end{bmatrix} = \begin{bmatrix}p_1 + td_1\\p_2 + td_2\end{bmatrix}$$

Comparing entries, we get **parametric equations** of the line:

$$\begin{cases} x_1 = p_1 + td_1 \\ x_2 = p_2 + td_2, \end{cases} \quad t \in \mathbb{R}$$

The familiar **scalar equation** of the line is obtained by eliminating the parameter t. Provided that $d_1 \neq 0$ we solve the first equation for t to get

$$\frac{x_1 - p_1}{d_1} = t$$

Substituting this into the second equation gives the scalar equation

$$x_2 = p_2 + \frac{d_2}{d_1}(x_1 - p_1) \tag{1.1}$$

What can you say about the line if $d_1 = 0$?

EXAMPLE 1.1.6

Write a vector equation, a scalar equation, and parametric equations of the line passing through the point $P(3,4)$ with direction vector $\begin{bmatrix}-5\\1\end{bmatrix}$.

Solution: A vector equation is $\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}3\\4\end{bmatrix} + t\begin{bmatrix}-5\\1\end{bmatrix}, t \in \mathbb{R}$.

So, parametric equations are $\begin{cases} x_1 = 3 - 5t \\ x_2 = 4 + t, \end{cases} \quad t \in \mathbb{R}$.

Hence, a scalar equation is $x_2 = 4 - \frac{1}{5}(x_1 - 3)$.

Directed Line Segments

For dealing with certain geometrical problems, it is useful to introduce **directed line segments**. We denote the directed line segment from point P to point Q by $\vec{PQ}$ as in Figure 1.1.6. We think of it as an "arrow" starting at P and pointing towards Q. We shall identify directed line segments from the origin O with the corresponding vectors; we write $\vec{OP} = \vec{p}$, $\vec{OQ} = \vec{q}$, and so on. A directed line segment that starts at the origin is called the **position vector** of the point.

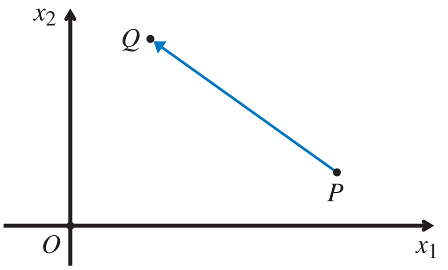


Figure 1.1.6 The directed line segment $\vec{PQ}$ from P to Q.

For many problems, we are interested only in the direction and length of the directed line segment; we are not interested in the point where it is located. For example, in Figure 1.1.3 on page 4, we may wish to treat the line segment $\vec{QR}$ as if it were the same as $\vec{OP}$. Taking our cue from this example, for arbitrary points P, Q, R in $\mathbb{R}^2$, we define $\vec{QR}$ to be **equivalent** to $\vec{OP}$ if $\vec{r} - \vec{q} = \vec{p}$. In this case, we have used one directed line segment $\vec{OP}$ starting from the origin in our definition.

More generally, for arbitrary points Q, R, S, and T in $\mathbb{R}^2$, we define $\vec{QR}$ to be equivalent to $\vec{ST}$ if they are both equivalent to the same $\vec{OP}$ for some P. That is, if

$$\vec{r} - \vec{q} = \vec{p} \text{ and } \vec{t} - \vec{s} = \vec{p} \text{ for the same } \vec{p}$$

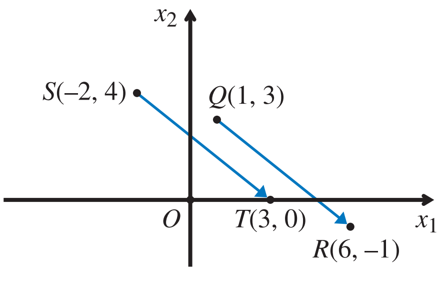
We can abbreviate this by simply requiring that

$$\vec{r} - \vec{q} = \vec{t} - \vec{s}$$

EXAMPLE 1.1.7

For points $Q(1,3)$, $R(6,-1)$, $S(-2,4)$, and $T(3,0)$, we have that $\vec{QR}$ is equivalent to $\vec{ST}$ because

$$\vec{r} - \vec{q} = \begin{bmatrix} 6 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} - \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \vec{t} - \vec{s}$$



In some problems, where it is not necessary to distinguish between equivalent directed line segments, we "identify" them (that is, we treat them as the same object) and write $\vec{PQ} = \vec{RS}$. Indeed, we identify them with the corresponding line segment starting at the origin, so in Example 1.1.7 we write $\vec{QR} = \vec{ST} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$.

Remark

Writing $\vec{QR} = \vec{ST}$ is a bit sloppy—an abuse of notation—because $\vec{QR}$ is not really the same object as $\vec{ST}$. However, introducing the precise language of "equivalence classes" and more careful notation with directed line segments is not helpful at this stage. By introducing directed line segments, we are encouraged to think about vectors that are located at arbitrary points in space. This is helpful in solving some geometrical problems, as we shall see below.

EXAMPLE 1.1.8

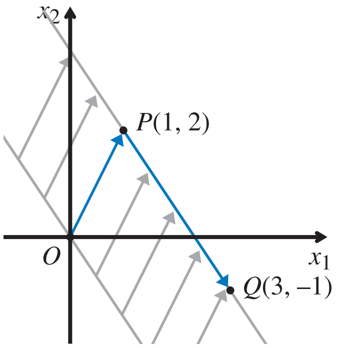
Find a vector equation of the line through $P(1, 2)$ and $Q(3, -1)$.

Solution: A direction vector of the line is

$$\vec{PQ} = \vec{q} - \vec{p} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Hence, a vector equation of the line with direction $\vec{PQ}$ that passes through $P(1, 2)$ is

$$\vec{x} = \vec{p} + t\vec{PQ} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad t \in \mathbb{R}$$



Observe in the example above that we would have the same line if we started at the second point and "moved" toward the first point—or even if we took a direction vector in the opposite direction. Thus, the same line is described by the vector equations

$$\vec{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + r\begin{bmatrix} -2 \\ 3 \end{bmatrix}, \quad r \in \mathbb{R}$$

$$\vec{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + s\begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad s \in \mathbb{R}$$

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} -2 \\ 3 \end{bmatrix}, \quad t \in \mathbb{R}$$

In fact, there are infinitely many descriptions of a line: we may choose any point on the line, and we may use any non-zero scalar multiple of the direction vector.

EXERCISE 1.1.3

Find a vector equation of the line through $P(1, 1)$ and $Q(-2, 2)$.